ON AUSLANDER-REITEN QUIVERS OF FINITE GROUPS(Representation Theory of Finite Groups and Finite Dimensional Algebras)

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1. Introduction

Let G be a finite group and k a field of characteristic $p > 0$. Let $\Gamma_s(kG)$ be the stable Auslander-Reiten quiver of the group algebra kG. By Webb's Theorem, the tree class of a connected component Δ of $\Gamma_s(kG)$ is restricted. We summarize results from [W, O1, Bt1, E-S] on the graph structure of connected components of $\Gamma_s(kG)$.

Theorem 1.1([W], [O1], [Bt1], [E-S]). Let Δ be a connected component of $\Gamma_s(kG)$. Then the tree class of Δ is A_n, $\tilde{A}_{1,2}$, $\tilde{B}_3$, A_∞, B_∞, C_∞, D_∞ or A_∞^∞. If k is algebraically closed, then the tree class is not B_∞ or C_∞. Moreover if the tree class or the reduced graph of Δ is Euclidean, then the modules in Δ lie in a block whose defect group is a Klein four group $C_2 \times C_2$.

Moreover if Δ contains the trivial kG-module k, then the graph structure of Δ has been investigated [W, L, O1, E2].

Theorem 1.2([W], [L], [O1], [E2]). Let Δ_0 be the connected component containing the trivial kG-module k and T the tree class of Δ_0. Let P be a Sylow p-subgroup of G. Then;

(1) If P is cyclic, then $T = A_n$ for some n.

(2) If $P = C_2 \times C_2$ and $N_G(P) = C_G(P)$, then $T = \tilde{A}_{1,2}$.

(3) If $P = C_2 \times C_2$ and $N_G(P) \neq C_G(P)$ but k does not contain a primitive cube root of unity, then $T = \tilde{B}_3$.

(4) If P is a dihedral 2-group and neither (2) nor (3) holds, then $T = A_\infty^\infty$. Moreover if P is dihedral of order at least 8, then $\Delta_0 \cong ZA_\infty^\infty$.

(5) If P is a semidihedral 2-group, then $T = D_\infty$ and $\Delta_0 \cong ZD_\infty$.

(6) If P is a generalized quaternion 2-group, then $T = A_\infty$ and Δ_0 is a 2-tube.

(7) $T = A_\infty$ and $\Delta_0 \cong ZA_\infty$ otherwise.

Here we study a connected component of $\Gamma_s(kG)$ containing an indecomposable kG-module whose k-dimension is not divided by p. Suppose that M is an indecomposable kG-module and $p \nmid \dim_k M$. In Section 2, we will show that M lies in a connected component isomorphic to ZA_∞ if k is an algebraically closed field of odd characteristic and a Sylow p-subgroup of G is not cyclic. In Sections 3 and 4 we consider the situation where $p = 2$ and a Sylow 2-subgroup of G is dihedral of order at least 8 or semidihedral. In Section 5 we make some remarks on tensoring the component containing the trivial kG-module k with M.

The notation is almost standard. For an indecomposable non-projective kG-module W, we write $A(W)$ to denote the Auslander-Reiten sequence (AR-sequence) $0 \to \Omega^2 W \to m(W) \to W \to 0$

terminating at W, where Ω is the Heller operator. The symbol $\otimes$ denotes tensor product over the coefficient field k. For an exact sequence of kG-modules $S : 0 \to A \to B \to C \to 0$ and a kG-module W, we write $S \otimes W$ to denote the tensor sequence $0 \to A \otimes W \to B \otimes W \to C \otimes W \to 0$. For tensoring the AR-sequence with an indecomposable kG-module, see [A-C, B-C]. If an exact sequence of kG-modules S is of the form $0 \to \Omega^2 W \oplus U' \to m(W) \oplus U \oplus U' \to W \oplus U \to 0$, where W is an indecomposable non-projective kG-module, and U and U' are projective or 0, we say that S is the AR-sequence $A(W)$ *modulo projectives.* Concerning some basic facts and terminologies used here, we refer to [Bn], [F] and [G].

2. ZA_∞–Component

Throughout this section, we assume that

(#2) k is algebraically closed and a Sylow p-subgroup P of G is not cyclic, dihedral, semidihedral or generalized quaternion.

First of all, we show

Theorem 2.1. Suppose that Θ is a connected component of $\Gamma_s(kG)$ containing an indecomposable kG-module whose k-dimension is not divided by p. Then

(1) Θ is isomorphic to ZA_∞ or ZD_∞.

(2) If p is odd, then Θ is isomorphic to ZA_∞.

(3) All modules in Θ have the same vertex P.

Remark. The above (3) follows from [U, Theorem 4.3].

Let M be an indecomposable kG-module with a Sylow p-subgroup P of G as vertex, and let S be a P-source of M. Then $p \nmid \dim_k M$ if and only if $p \nmid \dim_k S$ from [B-C, Proposition 2.4].

Proposition 2.2. Let M be an indecomposable kG-module such that $p \nmid \dim_k M$, and let S be a P-source of M. Let Θ be the connected component of $\Gamma_s(kG)$ containing M, and let Ξ be the connected component of $\Gamma_s(kP)$ containing S. Then

(1) Θ is isomorphic to ZA_∞ if and only if Ξ is isomorphic to ZA_∞.

(2) M lies at the end of ZA_∞-component if and only if S lies at the end of ZA_∞-component.

(3) Suppose that Θ is isomorphic to ZA_∞ and M lies at the end of Θ. Let $M — M_2 — \cdots — M_n — \cdots$ is a maximal tree of Θ with an irreducible map $M_{n+1} \to M_n$ ($n \geq 1$). Then there is a P-source S_n of M_n $(n \geq 2)$ such that $S — S_2 — \cdots — S_n — \cdots$ is a maximal tree of Ξ with an irreducible map $S_{n+1} \to S_n$ ($n \geq 1$).

Now we give examples of indecomposable kG-modules lying at the ends of ZA_∞-components.

Proposition 2.3. Let M be an indecomposable kG-module whose k-dimension is not divided by p. Let Q be a proper subgroup of P. Suppose that M satisfies the following conditions (with respect to Q) ;

(1) The trivial kQ-module k is a direct summand of $(M\otimes M)\downarrow_Q$ with multiplicity one;

(2) If Q is generalized quaternion, then $\Omega^2 k \nmid (M\otimes M)\downarrow_Q$.

Then M lies at the end of ZA_∞-component.

Remark. The above condition (1) is equivalent to the following condition: (1') We have an indecomposable direct sum decomposition $N \oplus (\oplus_i W_i)$ of $M\downarrow_Q$, where $p \nmid \dim_k N$ and $p \mid \dim_k W_i$ for all i.

From Proposition 2.3, we have following

Example 2.4. (1) Suppose that p is odd. Let M be an indecomposable kG-module with vertex P and S a P-source of M. Suppose that $\dim_k S = 2$. Then M lies at the end of ZA_∞-component.
(2) Suppose that $p \neq 3$. Let M be an indecomposable kG-module with vertex P and S a P-source of M. Suppose that $\dim_k S = 3$. Then M lies at the end of ZA_∞-component.

Proof. There exists an element x of P such that x does not act on S trivially. Let $Q = \langle x \rangle$. Then S satisfies the conditions (with respect to Q) in Proposition 2.3.

Remark. In [E3], Erdmann proved that if k is algebraically closed and a p-group P is not cyclic, dihedral, semidihedral or generalized quaternion, then there are infinitely many kP-modules of dimension 2 or 3 lying at the ends of ZA_∞-components ([E3], Propositions 4.2 and 4.4.). Using this result, she consequently showed that for a block B over an algebraically closed field, the stable Auslander-Reiten quiver $\Gamma_s(B)$ has infinitely many components of the form ZA_∞ if a defect group of B is not cyclic, dihedral, semidihedral or generalized quaternion.

3. Dihedral 2-group

In this section we consider the following situation:

(#3) k is an algebraically closed field of characteristic 2 and a Sylow 2-subgroup P of G is dihedral of order at least 8.

Let Δ_0 be the connected component containing the trivial kG-module k. Then Δ_0 is isomorphic to ZA_∞^∞ by Theorem 1.2. It is known that all modules in Δ_0 are endotrivial kG-modules (see, e.g., [Bt2]). Hence the following holds.

Proposition 3.1. Assume (#3). Let M be an odd dimensional indecomposable kG-module. Let Θ be the connected component of $\Gamma_s(kG)$ containing M and Δ_0 the connected component containing k. Then Θ is isomorphic to ZA_∞^∞ and tensoring with M induces a graph isomorphism from Δ_0 onto Θ. Moreover all modules in Θ have the same vertex P.

4. Semidihedral 2-group

Throughout this section, we assume that

(#4) k is an algebraically closed field of characteristic 2 and a Sylow 2-subgroup P of G is semidihedral.

Let Λ_0 be the connected component of $\Gamma_s(kP)$ containing the trivial kP-module k. Then Λ_0 is isomorphic to ZD_∞ (see [E2, p.76, II. 10.7 Remark). Thus a part of Λ_0 is as follows for some indecomposable kG-modules H_2, H_3 and I.

$$\begin{tikzcd}
 & \Omega^2 k \arrow[dr] & & k \arrow[dr] & & {} \\
\Omega^2 H_2 \arrow[ur] \arrow[r] \arrow[dr] & \Omega^2 I \arrow[r] & H_2 \arrow[ur] \arrow[r] \arrow[dr] & I \arrow[r] & \Omega^2 H_2 \arrow[ur] \arrow[r] \arrow[dr] & {} \\
 & H_3 \arrow[ur] \arrow[dr] & & \Omega^2 H_3 \arrow[ur] \arrow[dr] & & {} \\
{} \arrow[ur] & & {} \arrow[ur] & & {} &
\end{tikzcd}$$

Let $P = \langle x, y ; x^2 = y^{2^{n-1}} = 1, y^x = y^{-1+2^{n-2}} \rangle$ and $\mathfrak{X} = \{<x>\}$. Then an $\mathfrak{X}$-projective cover resolution of k is $0 \to \Omega_{\mathfrak{X}} k \to (k\downarrow_{<x>})\uparrow^P \to k \to 0$, where $(k\downarrow_{<x>})\uparrow^P \to k$ is a canonical epimorphism and $\Omega_{\mathfrak{X}} k$ is its kernel. Concerning some basic facts on relative projective cover, we refer to [Kn, T, O2].

In [O2], Okuyama showed the following

Theorem 4.1[O2]. With the same assumption and notations as above,

(1) $I \cong \Omega(\Omega_{\mathfrak{X}} k)$ and I is an endotrivial kP-module.

(2) I is self-dual and odd dimensional.

(3) If I' is self-dual, odd dimensional and indecomposable, then

$I' \cong k$ or I.

Applying Theorem 4.1, we have

Lemma 4.2. Let S be an odd dimensional indecomposable kP-module. Then $S \nmid S\otimes I$.

If S is an odd dimensional indecomposable kP-module, then the projective-free part S' of $S\otimes I$ is odd dimensional indecomposable and $S \not\cong S'$ by Theorem 4.1 and Lemma 4.2. Moreover it follows that the projective-free part of $S\otimes H_2$ is indecomposable. Therefore the following holds.

Proposition 4.3. Let S be an odd dimensional indecomposable kP-module and Ξ the connected component of $\Gamma_s(kP)$ containing S. Then

(1) Ξ is isomorphic to ZD_∞.

(2) All indecomposable kP-modules in Ξ have the same vertex P.

Remark. The above (2) follows from [E1, Theorem A].

Let $\begin{array}{ccccccccc} k & — & H_2 & — & H_3 & — \cdots — & H_n & — & \cdots \\ & & | & & & & & & \\ & & I & & & & & & \end{array}$ be a maximal tree of Λ_0.

If S is an odd dimensional indecomposable kG-module, then the projective-free part S_n of $H_n\otimes S$ is indecomposable and the tensor sequence $A(H_n)\otimes S$ is the AR-sequence $A(S_n)$ modulo projectives. Hence the following holds.

Lemma 4.4. Let S be an odd dimensional indecomposable kP-module and Ξ the connected component of $\Gamma_s(kP)$ containing S. Then tensoring with S induces a graph isomorphism from Λ_0 onto Ξ.

Using [Ka1, Theorem and Ka2, Theorem], we obtain

Proposition 4.5. Let M be an odd dimensional indecomposable kG-module and Θ the connected component containing M. Let Δ_0 be the connected component containing the trivial kG-module k. Then
(1) Θ is isomorphic to ZD_∞ and tensoring with M induces a graph isomorphism from Δ_0 onto Θ.
(2) All indecomposable kG-modules in Θ have the same vertex P.

5. Remarks on tensoring with a certain module

Suppose that M is an indecomposable kG-module and $p \nmid \dim_k M$. Let Θ be the connected component of $\Gamma_s(kG)$ containing M and Δ_0 the connected component containing the trivial kG-module k. If a Sylow p-subgroup P of G is dihedral of order at least 8 or semidihedral, then tensoring with M induces a graph isomorphism from Δ_0 onto Θ as we have seen in Propositions 3.1 and 4.5.

In this section we consider on tensoring modules in Δ_0 with M under the same hypothesis as in Section 2. Throughout this section, we assume that

(#2) k is algebraically closed and a Sylow p-subgroup P of G is not cyclic, dihedral, semidihedral or generalized quaternion.

Hence the connected component Δ_0 of $\Gamma_s(kG)$ containing the trivial kG-module k is of the form ZA_∞ by Theorem 1.2.

Proposition 5.1. Suppose that M is indecomposable kG-module and $p \nmid \dim_k M$. Let Θ be the connected component of $\Gamma_s(kG)$ containing M. Let S be a P-source of M and Ξ the connected component of $\Gamma_s(kP)$ containing S. Suppose that Θ is isomorphic to ZA_∞ and M lies at the end of Θ. Then the following are equivalent.
(1) Tensoring with M induces a graph isomorphism from Δ_0 onto Θ.
(2) Tensoring with S induces a graph isomorphism from the connected component of $\Gamma_s(kP)$ containing the trivial kP-module k onto Ξ.

Note that the hypothesis of Proposition 5.1 implies that $\Xi \cong ZA_\infty$ and S lies at the end of Ξ by Proposition 2.2.

Example 5.2. Let M be a trivial source module with vertex P. Let Θ be the connected component of $\Gamma_s(kG)$ containing M. Then Θ is isomorphic to ZA_∞ and M lies at the end of Θ. Moreover tensoring with M induces a graph isomorphism from Δ_0 onto Θ.

We consider an indecomposable kG-module M lying at the end of its connected component Θ isomorphic to ZA_∞. In the following, we give conditions which imply that tensoring with M induces a graph isomorphism from Δ_0 onto Θ.

Proposition 5.3. Let M be an indecomposable kG-module with $p \nmid \dim_k M$, and let Θ be the connected component of $\Gamma_s(kG)$ containing M. Suppose that M lies at the end of Θ and $M \otimes M^* \cong k \oplus (\oplus_i W_i)$, where each W_i is indecomposable and $p \mid \dim_k W_i$. Then tensoring with M induces a graph isomorphism from Δ_0 onto Θ.

Example 5.4. Suppose that M is an endotrivial kG-module. Let Θ be the connected component containing M. Then M satisfies the condition in Proposition 5.5. Hence tensoring with M induces a graph isomorphism from Δ_0 onto Θ.

Remark. Without the assumption (#2), if M is an endotrivial kG-module, then tensoring with M induces a graph isomorphism from the connected component containing the trivial kG-module onto the connected component containing M (Bt2, Theorem 2.3]). For related results on endotrivial modules, see also [Bt2].

Proposition 5.5. Let M be an indecomposable kG-module with $p \nmid \dim_k M$, and let Θ be the connected component of $\Gamma_s(kG)$ containing M. Let Q be a proper subgroup of P. Suppose that M satisfies the conditions (with respect to Q) in Proposition 2.3. Then tensoring with M induces a graph isomorphism from Δ_0 onto Θ.

Example 5.6. (1) Suppose that p is odd. Let M be an indecomposable kG-module with vertex P and S a P-source of M. Suppose that $\dim_k S = 2$. Then tensoring with M induces a graph isomorphism from Δ_0 onto the connected component containing M.

(2) Suppose that $p = 2$. Let M be an indecomposable kG-module with vertex P and S a P-source of M. Suppose that $\dim_k S = 3$. Then tensoring with M induces a graph isomorphism from Δ_0 onto the connected component containing M.

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